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COMPUTATIONAL ASPECTS OF MULTILEVEL TRAJECTORY OPTIMIZATION

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Abstract

This paper presents new computational results in multilevel trajectory optimization. The original formulation of trajectory decomposition is extended and applied to a difficult multiple arc trajectory example, the low thrust interplanetary swingby problem. A numerical solution is obtained for this trajectory which is characterized by nonlinear, time variant differential equations and interior boundaries and discontinuities. The time domain decomposition of the trajectory is made at the boundaries between arc segments. A three level optimization hierarchy is employed to transform the first feasible trajectory iterate into a final solution trajectory. This example is characteristic of a large class of interplanetary swingby problems which defy solution by conventional computational methods because of multiple discontinuities, constraints, and severe numerical sensitivities. The multilevel approach appears to be effective in obtaining a solution to problems of this type when other more conventional methods are unsatisfactory.

1. Introduction

The concept of trajectory decomposition was first introduced by Bauman[1] as a technique for optimizing trajectories with intermediate discontinuities. Trajectory decomposition represents an extension of multilevel systems theory, which has been under development since the early 1960's. The original work in multilevel theory is due to Mesarovic[2,3] who was concerned with creating a general mathematical structure for the study of organizations and hierarchies. Bauman devised a two level procedure for optimization of discontinuous trajectories utilizing the concept of feasible decomposition, first proposed for static systems by Brosilow, Lasdon, and Pearson[4] and then for nonlinear dynamic systems by Macko and Pearson.[5]

Computational experience using trajectory decomposition has been limited. This problem unfortunately pervades all of multilevel systems theory despite its existence for more than a decade. The principal reason for a dearth of computational results is that while the multilevel concept is theoretically appealing for complex systems, the numerical solution of realistic problems is generally a formidable task. Thus, the multilevel literature abounds with theoretical papers and papers treating simple examples analytically, but only a few works presenting computational results have appeared (see, for example, Refs. 1, 6, 7, 8, 9, and 10).

This paper presents the numerical solution of a low thrust interplanetary swingby problem by means of trajectory decomposition. This example is representative of a variety of future unmanned missions for exploration of the solar system. Low thrust swingby trajectories are characterized by high initial value sensitivities, nonlinear dynamics, and (for a sphere of influence model) intermediate discontinuities. Despite the interest in such trajectories, the difficulty in obtaining numerical solutions has limited the results published to date. Several examples of computational results which attest to the difficulty of this problem are Refs. 11, 12, 13, and 14. Because of its complexity and high numerical sensitivities, the swingby problem provides an excellent test of the utility of the multilevel approach.

A general decomposition procedure and its resulting three level optimization structure are presented next. The low thrust swingby example is then formulated, and results are given for a computational example. Details of the variational analysis leading to the set of decomposed necessary conditions may be found in Refs. 9 and 15.

2. General Decomposition Procedure

Given a state space trajectory consisting of N distinguishable arcs, it is required to minimize the Bolza performance index

$$J = \phi^o\left[\underline{x}^1(t_o^1),\ t_o^1\right] + \sum_{i=1}^{N}\left\{\phi^i\left[\underline{x}^i(t_f^i), t_f^i\right] + \int_{t_o^i}^{t_f^i} F^i\left[\underline{x}^i(t^i),\ \underline{u}^i(t^i),\ t^i\right] dt^i\right\} \tag{1}$$

subject to the dynamic constraints

$$\dot{\underline{x}}^i = \underline{f}^i[\underline{x}^i\ (t^i),\ \underline{u}^i\ (t^i),\ t^i] \tag{2}$$

$$\underline{C}^i[\underline{x}^i\ (t^i), \underline{u}^i\ (t^i),\ t^i] = \underline{0} \tag{3}$$

$$\underline{D}^i[\underline{x}^i\ (t^i),\ \underline{u}^i\ (t^i),\ t^i] \geq \underline{0} \tag{4}$$

$$\underline{S}^i\left[\underline{x}^i\ (t^i),\ t^i\right] = 0 \tag{5}$$

$$\underline{T}^i\left[\underline{x}^i\ (t^i),\ t^i\right] \geq 0 \tag{6}$$

where $t^i \in \left[t_o^i,\ t_f^i\right]$, $i = 1, \ldots, N$, the arc boundary conditions

$$\underline{\psi}^o\left[\underline{x}^1(t_o^1),\ t_o^1\right] = \underline{0} \tag{7}$$

$$\underline{\psi}^i\left[\underline{x}^i\ (t_f^i),\ t_f^i\right] = \underline{0} \tag{8}$$

for $i = 1, \ldots, N$, and the arc coupling relations

$$\underline{\Psi}^i\left[\underline{x}^i\ (t_f^i),\ t_f^i;\ \underline{x}^{i+1}\ (t_o^{i+1}),\ t_o^{i+1}\right] = \underline{0} \tag{9}$$

for $i = 1, \ldots, N - 1$. Here $\underline{x}^i$ is an n vector of states, $\underline{u}^i$ is an m_i vector of controls, t^i is time, $t_o^1 < t_f^1 \leq t_o^2 < \ldots \leq t_o^N < t_f^N$, ϕ^i and F^i are scalar functions, $\underline{C}^i$ is p_i vector of control equalities, $\underline{D}^i$ is a q_i vector of control inequalities, $\underline{S}^i$ is an s_i vector of state equalities, $\underline{T}^i$ is an r_i vector of state inequalities, $\underline{\psi}^i$ is a v_i vector of arc terminal constraints, $v_i \leq n + 1$, and $\underline{\Psi}^i$ is an n + 1 vector of inter-arc coupling relations.

Necessary conditions for optimization are generated by appending constraints (2) through (9) to the performance index (1) as follows:

$$J^* = \Phi^o + \sum_{i=1}^{N} \underline{\rho}^i \cdot \underline{\Psi}^i$$

$$+ \sum_{i=1}^{N} \left[\Phi^i + \int_{t_o^i}^{t_f^i} (H^i - \underline{\lambda}^i \cdot \underline{\dot{x}}^i)\, dt^i \right] \tag{10}$$

where

$$\Phi^o = \phi^o(\underline{x}_o^1,\ t_o^1) + \underline{\nu}^o \cdot \psi^o(x_o^1,\ t_0^1) \tag{11}$$

and

$$\Phi^i = \phi^i(\underline{x}_f^i,\ t_f^i) + \underline{\nu}^i \cdot \psi^i(\underline{x}_f^i,\ t_f^i) \tag{12}$$

$$H^i = F^i(\underline{x}^i, \underline{u}^i, t^i) + \mathscr{I}^i[\underline{S}^i(\underline{x}^i, t^i)] + \mathscr{R}^i[\underline{T}^i(\underline{x}^i, t^i)]$$

$$+ \underline{\lambda}^i \cdot \underline{f}^i(\underline{x}^i, \underline{u}^i, t^i) + \underline{\eta}^i \cdot \underline{C}^i(\underline{x}^i, \underline{u}^i, t^i)$$

$$+ \underline{\theta}^i \cdot \left[\underline{D}^i - [\underline{\alpha}^i]^2\right] \tag{13}$$

For J^* to possess a minimum, it is necessary that dJ^* vanish for arbitrary variations in all its arguments. This requires the coefficient of each perturbation quantity to vanish. Equating these coefficients to zero results in a set of decomposed first order necessary conditions which may be applied to each arc and boundary. For a minimum time trajectory problem, the decomposed necessary conditions may be arranged in the three level structure shown in Figure 1, with the following groupings:

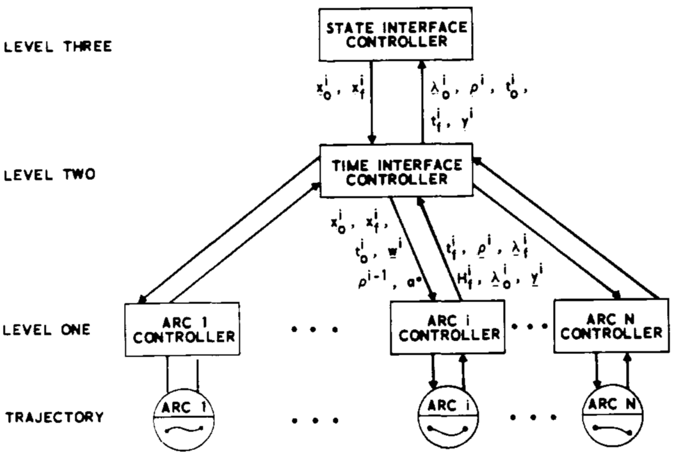

Figure 1. A Three Level Control Structure

Level One Necessary Conditions

$$\underline{f}^i - \underline{\dot{x}}^i = \underline{0} \tag{14}$$

$$\frac{\partial H^i}{\partial \underline{x}^i} + \underline{\dot{\lambda}}^i = \underline{0} \tag{15}$$

$$\frac{\partial H^i}{\partial \underline{u}^i} = \underline{0} \tag{16}$$

$$\underline{C}^i = \underline{0} \tag{17}$$

$$\left.\begin{aligned} D_j^i - \left[\alpha_j^i\right]^2 &= 0 \quad (18) \\ \theta_j^i \alpha_j^i &= 0 \quad (19) \end{aligned}\right\} \quad j = 1, \ldots, q_i$$

$$-\underline{\lambda}_f^i + \frac{\partial \phi^i}{\partial \underline{x}_f^i} + \underline{\nu}^i \cdot \frac{\partial \underline{\psi}^i}{\partial \underline{x}_f^i} + \underline{\rho}^i \cdot \frac{\partial \underline{\Psi}^i}{\partial \underline{x}_f^i} = \underline{0} \tag{20}$$

$$H_f^i + \frac{\partial \phi^i}{\partial t_f^i} + \underline{\nu}^i \cdot \frac{\partial \underline{\psi}^i}{\partial t_f^i} + \underline{\rho}^i \cdot \frac{\partial \underline{\Psi}^i}{\partial t_f^i} = 0 \tag{21}$$

$$\left.\begin{aligned} \underline{\psi}^i &= \underline{0} \quad (22) \\ \underline{\Psi}^i &= \underline{0} \quad (23) \end{aligned}\right\} \text{shared with levels two and three}$$

$$\frac{\partial^2 H^i}{(\partial \underline{u}^i)^2} \geq 0 \quad \text{(Clebsch Condition)} \tag{24}$$

for $i = 1, \ldots, N$, $\underline{\rho}^N = \underline{\Psi}^N \triangleq \underline{0}$.

Level Two Necessary Conditions

$$-H_o^1 + \frac{\partial \phi^o}{\partial t_o^1} + \underline{\nu}^o \cdot \frac{\partial \underline{\psi}^o}{\partial t_o^1} = 0 \tag{25}$$

$$-H_o^{i+1} + \underline{\rho}^i \cdot \frac{\partial \underline{\Psi}^i}{\partial t_o^{i+1}} = 0, \quad i = 1, \ldots, N-1 \tag{26}$$

plus appropriate components of Eqs. (22) and (23).

Level Three Necessary Conditions

$$\underline{\psi}^o = \underline{0} \tag{27}$$

$$\underline{\lambda}_o^1 + \frac{\partial \phi^o}{\partial \underline{x}_o^1} + \underline{\nu}^o \cdot \frac{\partial \underline{\psi}^o}{\partial \underline{x}_o^1} = \underline{0} \tag{28}$$

$$\underline{\lambda}_o^{i+1} + \underline{\rho}^i \cdot \frac{\partial \underline{\Psi}^i}{\partial \underline{x}_o^{i+1}} = 0, \quad i = 1, \ldots, N-1 \tag{29}$$

plus appropriate components of Eqs. (22) and (23).

The third level State Interface Controller adjusts the boundary conditions $\underline{x}_o^{i+1}$ in order to drive the transversality conditions Eqs. (28) and (29) to zero using information returned from below on the previous iteration. The second level Time Interface Controller coordinates the arc endpoint times, satisfies the intermediate time transversality conditions of Eq. (26), and performs adjoint variable rescaling to satisfy the terminal time transversality condition for the trajectory. The first level Arc i Controllers perform independent optimization of each arc subject to the boundary conditions imposed by the two higher level controllers. The iteration procedure is feasible in that each third level iteration results in a physically realizable (although generally non-optimim) solution trajectory.

3. A Low Thrust Swingby Example

The example to be considered is an Earth-Jupiter-Saturn continuous low thrust mission illustrated in Figure 2. A sphere of influence model is adopted in order that the spacecraft will be affected only by the gravity of a single central body along each arc. This approximation is usually adequate for mission planning purposes[16, 17] and results in a tractable dynamic model of the trajectory. A precise numerical definition of the so-called enlarged sphere of influence used here may be found in Ref. 17.

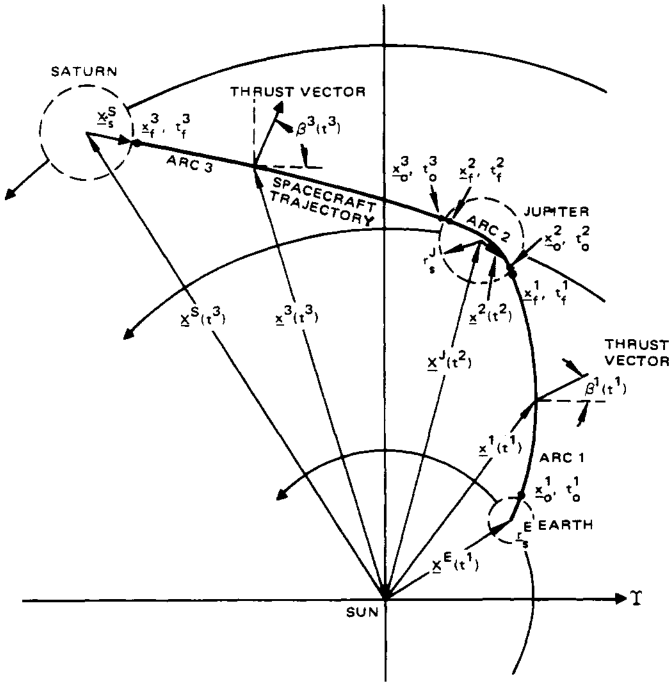

Figure 2. Geometry of the Low Thrust Interplanetary Swingby Problem

The state vector $\underline{x} \triangleq (x, y, u, v)^T$ consists of two position variables and two velocity variables measured in a cartesian frame. The control variable is $\beta(t)$, the time varying steering angle measured from the inertial direction ♈, the first point in Aries. The vector $\underline{X}^P(t) = [X^P(t), Y^P(t), U^P(t), V^P(t)]^T$ represents the state of planet P, where P is replaced by E for Earth, J for Juipter, and S for Saturn.

The sphere of influence model suggests an obvious decomposition of the trajectory into three arcs. Arc 1 will be defined as the heliocentric segment between the spheres of influence of Earth and Jupiter. Arc 2 is the planetocentric segment entirely within the Jupiter sphere. Arc 3 is the heliocentric segment from the Juipter sphere exit point to the sphere of influence of Saturn. The state variables and their derivatives are all discontinuous across the Jupiter sphere of influence because of the coordinate system switch and the change of the gravitational constant of the central body.

Mathematically, the problem is to select $\beta^i(t^i)$, $t^i \epsilon [t_o^i, t_f^i]$, i = 1, 2, 3 such that the duration of flight

$$J = t_f^3 - t_o^1 = \sum_{i=1}^{3} J^i = \sum_{i=1}^{3} \left[\int_{t_o^i}^{t_f^i} dt^i \right] \qquad (30),$$

is minimized. The differential equations governing the motion are

$$\left.\begin{aligned}
\dot{x}^i &= u^i && (31)\\
\dot{y}^i &= v^i && (32)\\
\dot{u}^i &= -\frac{\mu^i x^i}{(r^i)^3} + a^i(t^i)\cos\beta^i && (33)\\
\dot{v}^i &= -\frac{\mu^i x^i}{(r^i)^3} + a^i(t^i)\sin\beta^i && (34)
\end{aligned}\right\} i = 1, 2, 3$$

where

$$r^i = [(x^i)^2 + (y^i)^2]^{1/2} \qquad (35)$$

$$a^i(t^i) = \frac{cq}{(m_o^i - qt^i)} \qquad (36)$$

and c is a constant exhaust velocity, q is a constant mass explusion rate, m_o^i is the vehicle mass at t_o^i, $a^i(t^i)$ is the time varying thrust acceleration, and μ^i is the gravitational constant for the central body of arc i. The μ^1, μ^3 represent the Sun and μ^2 represents Jupiter.

Application of the necessary conditions Eqs. (14)—(29) to the above dynamics and associated sphere of influence entry and exit conditions produces a complete set of arc and boundary relations for this problem.[9] The next section presents computational aspects of the swingby problem.

4. Numerical Solution of the Example

Construction of the three level process for a given numerical problem requires the selection of specific algorithms to perform the various optimization tasks. The requirements for each level are as follows:

Level One

It is desirable to select an algorithm (or algorithms) which converges quickly from various initial regions, since each arc evaluation requires the complete integration of a set of differential equations. If an excessive number of function evaluations are needed, the frequent repetition of the first level task will use up most of the available computer time. If a first level algorithm fails to converge, the entire multilevel iteration procedures comes to a halt.

The optimization technique selected for each first level controller is a modification of the Marquardt-Levenberg maximum neighborhood method. The basic technique was first introduced by Levenberg[18] and later independently by Marquardt[19] in connection with the least squares estimation of nonlinear parameters. This mathematical programming approach has been applied recently to dynamic trajectory optimization problems by Starr and Sugar[20] and by Armstrong, Childs, and Markos.[21] Considerable computational experimentation by Wertz[22] has resulted in several modifications to Marquardt's original algorithm. These modifications, embodied in a subprogram GAUSAUS, render the basic algorithm more adaptive to irregular contour regions and also generally accelerate its convergence. A description of this algorithm is provided in Ref. 9.

Level Two

All necessary conditions assigned to the second level are satisfied directly and do not require the use of an iterative algorithm.

Level Three

With all lower level conditions satisfied, the third level must adjust $\underline{x}_o^{i+1}$ to satisfy Eqs. (27)—(29) and thereby minimize J as given in Eq. (30).

The gradient of the performance index remaining for the third level is

$$\underline{\nabla} J^* = \underline{\nabla} t_f^3 = \begin{bmatrix} \lambda_{yo}^2 - \lambda_{yf}^1 + \dfrac{\left[y_f^1 - Y^J(t_f^1)\right]}{\left[x_f^1 - X^J(t_f^1)\right]} \left(\lambda_{xf}^1 - \lambda_{xo}^2\right) \\ \lambda_{uo}^2 - \lambda_{uf}^1 \\ \lambda_{vo}^2 - \lambda_{uf}^1 \\ \lambda_{yo}^3 - \lambda_{yf}^2 + \dfrac{y_f^2}{x_f^2} \left(\lambda_{xf}^2 - \lambda_{xo}^3\right) \\ \lambda_{uo}^3 - \lambda_{uf}^2 \\ \lambda_{vo}^3 - \lambda_{vf}^2 \end{bmatrix} \tag{37}$$

where t_o^1 is fixed. Equation (37) is used in conjunction with Eq. (30) each iteration to select new values for y_o^2, u_o^2, v_o^2, y_o^3, u_o^3, v_o^3, by means of a standard gradient algorithm. Details of the procedure are given in Ref. 9.

First Feasible Trajectory

A first feasible swingby trajectory was constructed after considerable numerical experimentation. The construction process was guided by various physical characteristics known for free fall swingby trajectories[17] and by previous experience with single arc low thrust trajectories.[20] The first feasible trajectory satisfies all but the third level adjoint conditions (i.e., the sphere of influence entry and exit point transversality conditions), and thus represents a physically realizable (non-optimum) solution. The value of the performance index t_f^3 is 2.73 years, representing a high energy interplanetary mission. The numerical characteristics of the trajectory are given in Table 1, with distances in units of astronomical units, time in years, and velocity in au/yr. The interplanetary probe has an initial thrust to weight ratio of 2.1 x 10^{-5}, reflecting typical future low thrust capability.

Three Level Iteration Sequence

The first feasible trajectory was used as the initial iterate for the three level optimization procedure. The behavior of the performance $J = t_f^3$ with level three iteration number is shown in Figure 3.

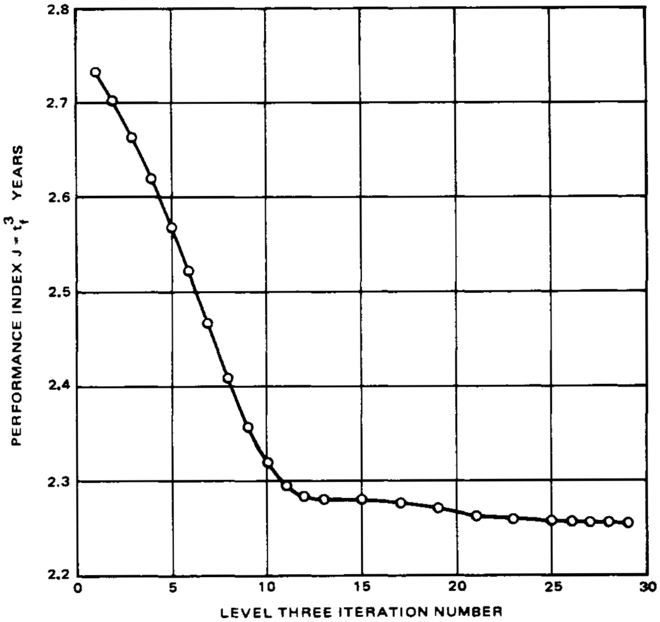

Figure 3. Behavior of the Performance Index with Level Three Iteration

If an iteration was successful, the step size control employed in the third level gradient algorithm was increased by 20 percent. This cautious increase was necessary to prevent convergence failures in the resulting arc optimizations. If a step failed to produce an improvement in J, or caused a lower level convergence failure, the step size control was reduced by half until an acceptable step was found.

As indicated in Figure 3, the improvement in J became very small beyond the 25th iteration. The major gains were achieved on the first 11 iterations. The procedure was terminated at iteration 29.

The final trajectory is shown in Figure 4. The thrust direction along each arc is indicated by the arrows. The value of $J = t_f^3$ is 2.26 years, which represents a reduction of 0.47 years in flight time and a propellant savings of 86.7 kg based upon an initial spacecraft mass of 1000.0 kg. The final trajectory is characterized by a sharper turn angle at Jupiter, giving it more energy at the swingby, and by a Jupiter-Saturn leg which curves away from the sun under the power of the vehicle's thrust in order to intercept the Saturn sphere of influence earlier in the planet's orbit. Computer time averaged 41 seconds on CDC 6600 equipment for each level three iteration. The majority of this time was spent in numerical integration of the trajectory arcs.

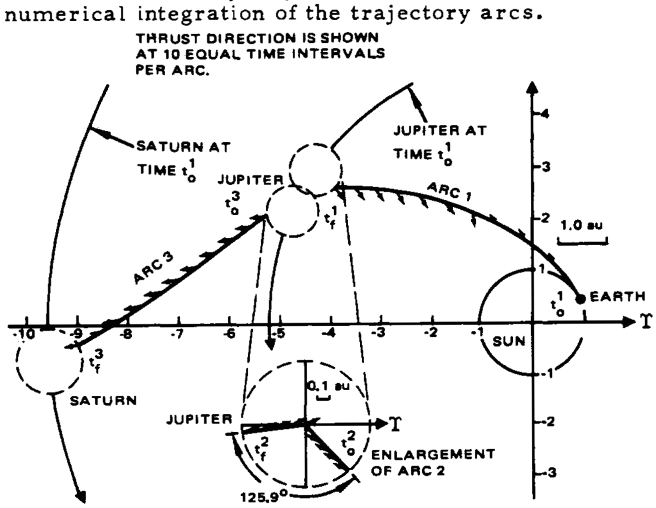

Figure 4. The Final Trajectory

Table 1. Numerical Data for the First Feasible Trajectory

		$\underline{x}^i = \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix}$	$\underline{X}^P = \begin{bmatrix} X \\ Y \\ U \\ V \end{bmatrix}$	$\underline{\lambda}^i = \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_u \\ \lambda_v \end{bmatrix}$	H^i
ARC 1	$t_o^i = 0.0$	9.52145480E-1 2.56416313E-1 -2.31714952E0 8.36648670E0	Earth 9.66566570E-1 2.56416313E-1 -1.61111090E0 6.07311570E0	-3.80924957E-3 -3.89273446E-3 5.51264121E-4 -1.74484653E-3	9.70600657E-1
	$t_f^i = 1.02913429$	-3.90451175E0 2.61785051E0 -3.48085778E0 -1.61022010E-1	Jupiter -4.25944841E0 2.98726299E0 -1.58171263E0 -2.25531645E0	4.74113327E-3 -5.43969528E-3 -2.62564925E-3 3.36229127E-3	9.70048209E-1
ARC 2	$t_o^2 = 1.02913429$	3.54936662E-1 -3.69412482E-1 -1.89914515E0 2.09429444E0	$\underline{X}^J(t_o^2) = \underline{X}^J(t_f^1)$, shown above.	4.89031801E-3 -8.02677113E-3 7.18105460E-4 -1.29977986E-3	9.71230423E-1
	$t_f^2 = 1.37138948$	-4.45050300E-1 2.53723957E-1 -2.52432036E0 1.20738972E0	$\underline{X}^J(t_f^2) = \underline{X}^J(t_o^3)$, shown below.	5.23543157E-3 -9.89549328E-3 -9.36751255E-4 1.69731152E-3	9.71111352E-1
ARC 3	$t_o^3 = 1.37138948$	-5.17314013E0 2.42439379E0 -3.67365870E0 -1.29606560E0	Jupiter -4.72808983E0 2.17066983E0 -1.14933834E0 -2.50345532E0	-2.53909574E-1 5.47498195E-1 -2.48473834E-1 7.73707574E-1	-9.07906753E-1
	$t_f^3 = 2.73092720$	-8.86260383E0 -1.35626032E0 -1.84477434E0 -4.33680038E0	Saturn -9.40751800E0 -1.67086666E0 3.55461771E-1 -2.0013648E0	-1.63432643E-1 5.82156996E-1 4.82072010E-9 -2.98084076E-10	-1.22320233E0

5. Conclusions

This paper has demonstrated the feasibility of a trajectory decomposition technique for numerical solution of a difficult multiple arc trajectory example. A multilevel formulation for minimum time trajectories was presented and certain computational aspects of the example problem were investigated. The principal conclusion from this study is that use of hierarchial techniques for the optimization of trajectories is a useful approach when other more conventional methods prove inadequate.

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